

Ramsey numbers for trees II

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Abstract

Let $r(G_1, G_2)$ be the Ramsey number of the two graphs G_1 and G_2 . For $n_1 \geq n_2 \geq 1$ let $S(n_1, n_2)$ be the double star given by $V(S(n_1, n_2)) = \{v_0, v_1, \dots, v_{n_1}, w_0, w_1, \dots, w_{n_2}\}$ and $E(S(n_1, n_2)) = \{v_0v_1, \dots, v_0v_{n_1}, v_0w_0, w_0w_1, \dots, w_0w_{n_2}\}$. In this paper we determine $r(K_{1,m-1}, S(n_1, n_2))$ for $n_1 \geq m-2 \geq n_2$. For $n \geq 6$ let $T_n^3 = S(n-5, 3)$, $T_n'' = (V, E_2)$ and $T_n''' = (V, E_3)$, where $V = \{v_0, v_1, \dots, v_{n-1}\}$, $E_2 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_2v_{n-1}\}$ and $E_3 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_2v_{n-2}, v_3v_{n-1}\}$. In this paper we obtain explicit formulas for $r(K_{1,m-1}, T_n)$ ($n > m+3$), $r(T_m', T_n)$ ($n > m+4$), $r(T_n, T_n)$, $r(T_n', T_n)$ and $r(P_n, T_n)$, where $T_n \in \{T_n'', T_n''', T_n^3\}$, P_n is the path on n vertices and T_n' is the unique tree with n vertices and maximal degree $n-2$.

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1. Introduction

In this paper, all graphs are simple graphs. For a graph $G = (V(G), E(G))$ let $e(G) = |E(G)|$ be the number of edges in G , and let $\Delta(G)$ and $\delta(G)$ denote the maximal degree and minimal degree of G , respectively.

For a graph G , as usual \overline{G} denotes the complement of G . Let G_1 and G_2 be two graphs. The Ramsey number $r(G_1, G_2)$ is the smallest positive integer n such that, for every graph G with n vertices, either G contains a copy of G_1 or else \overline{G} contains a copy of G_2 .

Let \mathbb{N} be the set of positive integers. For $n \in \mathbb{N}$ with $n \geq 6$ let T_n be a tree on n vertices. As mentioned in [R], recently Zhao proved the following conjecture of Burr and Erdős ([BE]): $r(T_n, T_n) \leq 2n-2$.

Let $m, n \in \mathbb{N}$. For $n \geq 3$ let $K_{1,n-1}$ denote the unique tree on n vertices with $\Delta(K_{1,n-1}) = n-1$, and for $n \geq 4$ let T_n' denote the unique tree on n vertices with $\Delta(T_n') = n-2$. In 1973 Burr and Roberts [BR] showed that for $m, n \geq 3$,

$$(1.1) \quad r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m+n-3 & \text{if } 2 \nmid mn, \\ m+n-2 & \text{if } 2 \mid mn. \end{cases}$$

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In 1995, Guo and Volkmann [GV] proved that for $n > m \geq 4$,

$$(1.2) \quad r(K_{1,m-1}, T'_n) = \begin{cases} m+n-3 & \text{if } 2 \mid m(n-1), \\ m+n-4 & \text{if } 2 \nmid m(n-1). \end{cases}$$

In 2012 the author [S] evaluated the Ramsey number $r(T_m, T_n^*)$ for $T_m \in \{P_m, K_{1,m-1}, T'_m, T_m^*\}$, where P_m is a path on m vertices and T_n^* is the tree on n vertices with $V(T_n^*) = \{v_0, v_1, \dots, v_{n-1}\}$ and $E(T_n^*) = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In particular, he proved that for $n > m \geq 7$,

$$(1.3) \quad r(K_{1,m-1}, T_n^*) = \begin{cases} m+n-3 & \text{if } m-1 \mid n-3, \\ m+n-4 & \text{if } m-1 \nmid n-3. \end{cases}$$

For $n \geq 5$ let $T_n^1 = (V, E_1)$ and $T_n^2 = (V, E_2)$ be the trees on n vertices with $V = \{v_0, v_1, \dots, v_{n-1}\}$, $E_1 = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-4}v_{n-2}, v_{n-3}v_{n-1}\}$ and $E_2 = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. Then $\Delta(T_n^1) = \Delta(T_n^2) = \Delta(T_n^*) = n-3$. In [SWW] the author and his coauthors proved that

$$(1.4) \quad r(K_{1,m-1}, T_n^1) = r(K_{1,m-1}, T_n^2) = m+n-4 \quad \text{for } n > m \geq 7 \quad \text{and } 2 \mid mn.$$

For $n_1, n_2 \in \mathbb{N}$ with $n_1 \geq n_2$ let $S(n_1, n_2)$ be the double star given by

$$\begin{aligned} V(S(n_1, n_2)) &= \{v_0, v_1, \dots, v_{n_1}, w_0, w_1, \dots, w_{n_2}\}, \\ E(S(n_1, n_2)) &= \{v_0v_1, \dots, v_0v_{n_1}, v_0w_0, w_0w_1, \dots, w_0w_{n_2}\}. \end{aligned}$$

Then clearly $T'_n = S(n-3, 1)$ and $T_n^2 = S(n-4, 2)$. In this paper we prove the following general result:

$$(1.5) \quad r(K_{1,m-1}, S(n_1, n_2)) = \begin{cases} m+n_1 & \text{if } n_1 \geq m-2 \geq n_2 \geq 2, 2 \mid mn_1 \\ & \text{and } n_1 > m-5+n_2 + \frac{(n_2-1)(n_2-2)}{m-1-n_2}, \\ m-1+n_1 & \text{if } n_1 \geq m-2 > n_2, 2 \nmid mn_1 \\ & \text{and } n_1 > m-5+n_2 + \frac{(n_2-1)^2}{m-2-n_2}. \end{cases}$$

We also prove that

$$(1.6) \quad r(K_{1,m-1}, T_n^1) = m+n-5 \quad \text{for } n > m \geq 6 \quad \text{and } 2 \nmid mn.$$

For $n \geq 6$ let $T_n^3 = S(n-5, 3)$, $T_n'' = (V, E_2)$ and $T_n''' = (V, E_3)$, where

$$\begin{aligned} V &= \{v_0, v_1, \dots, v_{n-1}\}, \quad E_2 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_2v_{n-1}\}, \\ E_3 &= \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_2v_{n-2}, v_3v_{n-1}\}. \end{aligned}$$

Then $\Delta(T_n^3) = \Delta(T_n'') = \Delta(T_n''') = n-4$. In this paper we evaluate $r(K_{1,m-1}, T_n)$ and $r(T'_m, T_n)$ for $T_n \in \{T_n^3, T_n'', T_n'''\}$. In particular, we show that for $n \geq 15$ and $m \geq 7$,

$$(1.7) \quad \begin{aligned} r(K_{1,m-1}, T_n'') &= r(K_{1,m-1}, T_n''') \\ &= \begin{cases} m+n-5 & \text{if } 2 \mid m(n-1) \text{ and } n > m+1 + \frac{8}{m-6}, \\ m+n-6 & \text{if } 2 \nmid m(n-1) \text{ and } n > m+3 + \frac{4}{m-5}, \end{cases} \end{aligned}$$

and that for $m \geq 9$ and $n > m + 1 + \max\{3, \frac{11}{m-8}\}$,

$$(1.8) \quad r(T'_m, T''_n) = (T'_m, T'''_n) = r(T'_m, T_n^3) = \begin{cases} m+n-5 & \text{if } m-1 \mid n-5, \\ m+n-6 & \text{if } m-1 \nmid n-5. \end{cases}$$

We also prove that for $n \geq 15$, $m \geq 9$, $m-1 \nmid n-5$ and $n \geq (m-3)^2 + 4$,

$$(1.9) \quad r(G_m, T_n) = m+n-6 \quad \text{for } G_m \in \{T_m^*, T_m^1, T_m^2\} \text{ and } T_n \in \{T''_n, T'''_n, T_n^3\}.$$

In addition, we establish the following results:

$$\begin{aligned} r(T''_n, T''_n) &= r(T''_n, T'''_n) = r(T'''_n, T'''_n) = \begin{cases} 2n-9 & \text{if } 2 \mid n \text{ and } n \geq 10, \\ 2n-8 & \text{if } 2 \nmid n \text{ and } n > 22, \end{cases} \\ r(T_n^3, T''_n) &= r(T_n^3, T'''_n) = r(T_n^3, T_n^3) = 2n-8 \quad \text{for } n > 22, \\ r(T''_n, T'_n) &= r(T'''_n, T'_n) = r(T_n^3, T'_n) = 2n-5 \quad \text{for } n \geq 10, \\ r(T''_n, T_n^i) &= r(T'''_n, T_n^i) = r(T_n^3, T_n^i) = 2n-7 \quad \text{for } n > 16 \text{ and } i = 1, 2, \\ r(P_n, T''_n) &= r(P_n, T'''_n) = r(P_n, T_n^3) = 2n-9 \quad \text{for } n \geq 33. \end{aligned}$$

In addition to the above notation, throughout this paper we also use the following notation: $[x]$ —the greatest integer not exceeding x , $d(v)$ —the degree of the vertex v in a graph, $d(u, v)$ —the distance between the two vertices u and v in a graph, K_n —the complete graph on n vertices, $G[V_1]$ —the subgraph of G induced by vertices in the set V_1 , $G - V_1$ —the subgraph of G obtained by deleting vertices in V_1 and all edges incident with them, $\Gamma(v)$ —the set of vertices adjacent to the vertex v , $\Gamma_2(v)$ —the set of those vertices u such that $d(u, v) = 2$, $e(V_1 V'_1)$ —the number of edges with one endpoint in V_1 and another endpoint in V'_1 ,

2. Basic lemmas

For a forbidden graph L let $ex(p; L)$ be the maximal number of edges in a graph of order p not containing any copies of L . The corresponding Turán's problem is to evaluate $ex(p; L)$. Let $p, n \in \mathbb{N}$ with $p \geq n \geq 2$. For a given tree T_n on n vertices, it is difficult to determine the value of $ex(p; T_n)$. The famous Erdős-Sós conjecture asserts that $ex(p; T_n) \leq \frac{(n-2)p}{2}$. Write $p = k(n-1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. In 1975 Faudree and Schelp [FS] showed that

$$(2.1) \quad ex(p; P_n) = k \binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}.$$

In [SW,SWW,ST] the author and his coauthors determined $ex(p; T_n)$ for $T_n \in \{T'_n, T_n^*, T_n^1, T_n^2, T_n^3, T''_n, T'''_n\}$.

Lemma 2.1 ([S, Lemma 2.1]). *Let G_1 and G_2 be two graphs. Suppose $p \in \mathbb{N}$, $p \geq \max\{|V(G_1)|, |V(G_2)|\}$ and $ex(p; G_1) + ex(p; G_2) < \binom{p}{2}$. Then $r(G_1, G_2) \leq p$.*

Proof. Let G be a graph of order p . If $e(G) \leq ex(p; G_1)$ and $e(\overline{G}) \leq ex(p; G_2)$, then $ex(p; G_1) + ex(p; G_2) \geq e(G) + e(\overline{G}) = \binom{p}{2}$. This contradicts the assumption. Hence, either $e(G) > ex(p; G_1)$ or $e(\overline{G}) > ex(p; G_2)$. Therefore, G contains a copy of G_1 or \overline{G} contains a copy of G_2 . This shows that $r(G_1, G_2) \leq |V(G)| = p$. So the lemma is proved.

Lemma 2.2. *Let $k, p \in \mathbb{N}$ with $p \geq k + 1$. Then there exists a k -regular graph of order p if and only if $2 \mid kp$.*

This is a known result, see for example [SW, Corollary 2.1].

Lemma 2.3 ([S, Lemma 2.3]). *Let G_1 and G_2 be two graphs with $\Delta(G_1) = d_1 \geq 2$ and $\Delta(G_2) = d_2 \geq 2$. Then*

- (i) $r(G_1, G_2) \geq d_1 + d_2 - (1 - (-1)^{(d_1-1)(d_2-1)})/2$.
- (ii) *Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \leq m$. Then $r(G_1, G_2) \geq 2d_2 - 1 \geq d_1 + d_2$.*
- (iii) *Suppose that G_1 is a connected graph of order m and $d_2 > m$. If one of the conditions*

- (1) $2 \mid (d_1 + d_2 - m)$,
- (2) $d_1 \neq m - 1$,
- (3) G_2 has two vertices u and v such that $d(v) = \Delta(G_2)$ and $d(u, v) = 3$

holds, then $r(G_1, G_2) \geq d_1 + d_2$.

Lemma 2.4. *Let $p, n \in \mathbb{N}$ with $p \geq n - 1 \geq 1$. Then $ex(p; K_{1, n-1}) = \lfloor \frac{(n-2)p}{2} \rfloor$.*

See for example [SW, Theorem 2.1].

Lemma 2.5 ([SW, Theorem 3.1]). *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 5$. Let $r \in \{0, 1, \dots, n - 2\}$ be given by $p \equiv r \pmod{n - 1}$. Then*

$$ex(p; T'_n) = \begin{cases} \lfloor \frac{(n-2)(p-1) - r - 1}{2} \rfloor & \text{if } n \geq 7 \text{ and } 2 \leq r \leq n - 4, \\ \frac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.6 ([SWW, Theorems 2.1 and 3.1]). *Suppose $p, n \in \mathbb{N}$, $p \geq n - 1 \geq 4$ and $p = k(n - 1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n - 2\}$. Then for $i = 1, 2$ we have*

$$\begin{aligned} ex(p; T_n^i) &= \max \left\{ \left\lfloor \frac{(n-2)p}{2} \right\rfloor - (n-1+r), \frac{(n-2)p - r(n-1-r)}{2} \right\} \\ &= \begin{cases} \left\lfloor \frac{(n-2)p}{2} \right\rfloor - (n-1+r) & \text{if } n \geq 16 \text{ and } 3 \leq r \leq n-6 \text{ or if} \\ & 13 \leq n \leq 15 \text{ and } 4 \leq r \leq n-7, \\ \frac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 2.7 ([ST, Theorems 3.1 and 5.1]). *Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$, $p = k(n - 1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n - 2\}$. Then*

$$ex(p; T_n'') = ex(p; T_n''') = \frac{(n-2)p - r(n-1-r)}{2} + \max \left\{ 0, \left\lfloor \frac{r(n-4-r) - 3(n-1)}{2} \right\rfloor \right\}.$$

Lemma 2.8 ([ST, Lemmas 4.6 and 4.7]). *Let $n \in \mathbb{N}$ with $n \geq 15$. Then*

$$ex(2n - 9; T_n^3) = n^2 - 10n + 24 + \max \left\{ \left\lfloor \frac{n}{2} \right\rfloor, 13 \right\}$$

and

$$ex(2n - 8; T_n^3) = n^2 - 9n + 29 + \max \left\{ \left\lfloor \frac{n-37}{4} \right\rfloor, 0 \right\}.$$

Lemma 2.9 ([ST, Theorems 4.1-4.5]). *Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$, $p = k(n - 1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n - 2\}$.*

- (i) If $r \in \{0, 1, 2, n-6, n-5, n-4, n-3, n-2\}$, then $\text{ex}(p; T_n^3) = \frac{(n-2)p-r(n-1-r)}{2}$.
(ii) If $n \geq 15$ and $r \in \{3, 4, \dots, n-9\}$, then

$$\text{ex}(p; T_n^3) = \frac{(n-2)p-r(n-1-r)}{2} + \max \left\{ 0, \left\lceil \frac{r(n-4-r)-3(n-1)}{2} \right\rceil \right\}.$$

- (iii) If $n \geq 15$ and $r = n-8$, then

$$\text{ex}(p; T_n^3) = \frac{(n-2)p-7n+30}{2} + \max \left\{ \left\lceil \frac{n}{2} \right\rceil, 13 \right\}.$$

- (iv) If $n \geq 15$ and $r = n-7$, then

$$\text{ex}(p; T_n^3) = \frac{(n-2)p-6(n-7)}{2} + \max \left\{ \left\lceil \frac{n-37}{4} \right\rceil, 0 \right\}.$$

Lemma 2.10. Let $n \in \mathbb{N}$, $n \geq 10$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Assume $p = k(n-1) + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then

$$\text{ex}(p; T_n) \leq \frac{(n-2)p}{2} - \min \left\{ n-1+r, \frac{r(n-1-r)}{2} \right\}.$$

Proof. This is immediate from [ST, Lemmas 2.8, 3.1, 4.1 and 5.1].

Lemma 2.11 ([SW, Theorems 4.1-4.3]). Let $p, n \in \mathbb{N}$, $p \geq n \geq 6$ and $p = k(n-1) + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, n-5, n-4, n-3, n-2\}$. Then

$$\text{ex}(p; T_n^*) = \begin{cases} \frac{(n-2)(p-2)}{2} + 1 & \text{if } n > 6 \text{ and } r = n-5, \\ \frac{(n-2)p-r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.12 ([SW, Theorem 4.4]). Let $p, n \in \mathbb{N}$, $p \geq n \geq 11$, $r \in \{2, 3, \dots, n-6\}$ and $p \equiv r \pmod{n-1}$. Let $t \in \{0, 1, \dots, r+1\}$ be given by $n-3 \equiv t \pmod{r+2}$. Then

$$\text{ex}(p; T_n^*) = \begin{cases} \left\lceil \frac{(n-2)(p-1)-2r-t-3}{2} \right\rceil & \text{if } r \geq 4 \text{ and } 2 \leq t \leq r-1, \\ \frac{(n-2)(p-1)-t(r+2-t)-r-1}{2} & \text{otherwise.} \end{cases}$$

3. Formulas for $r(T_n, T_n'')$, $r(T_n, T_n''')$ and $r(T_n, T_n^3)$

Theorem 3.1. Let $n \in \mathbb{N}$ with $n \geq 10$. Then

$$r(T_n'', T_n'') = r(T_n'', T_n''') = r(T_n''', T_n''') = \begin{cases} 2n-9 & \text{if } 2 \mid n \\ 2n-8 & \text{if } 2 \nmid n \text{ and } n > 22. \end{cases}$$

Proof. Let $T_n, T_n^0 \in \{T_n'', T_n'''\}$. By Lemma 2.7, we have $\text{ex}(2n-9; T_n) = \text{ex}(2n-9; T_n^0) = \left\lceil \frac{(2n-9)(n-5)}{2} \right\rceil$ and $\text{ex}(2n-8; T_n) = \text{ex}(2n-8; T_n^0) = n^2 - 9n + 29$. When $2 \mid n$ we have

$$\text{ex}(2n-9; T_n) + \text{ex}(2n-9; T_n^0) = 2 \left\lceil \frac{(2n-9)(n-5)}{2} \right\rceil = (2n-9)(n-5) - 1 < \binom{2n-9}{2}.$$

Thus, by Lemma 2.1 we have $r(T_n, T_n^0) \leq 2n - 9$. By Lemma 2.3(i),

$$r(T_n, T_n^0) \geq n - 4 + n - 4 - \frac{1 - (-1)^{(n-5)(n-5)}}{2} = 2n - 9.$$

Hence $r(T_n, T_n) = 2n - 9$ for even n .

Now assume that $2 \nmid n$. By Lemma 2.2, we may construct a regular graph G of order $2n - 9$ with degree $n - 5$. Clearly \overline{G} is also a regular graph with degree $n - 5$. Hence both G and \overline{G} do not contain any copies of T_n . Therefore $r(T_n, T_n^0) > 2n - 9$. On the other hand, for $n > 22$,

$$\text{ex}(2n - 8; T_n) + \text{ex}(2n - 8; T_n^0) = 2(n^2 - 9n + 29) < 2n^2 - 17n + 36 = \binom{2n - 8}{2}.$$

Hence $r(T_n, T_n^0) \leq 2n - 8$ by Lemma 2.1. Thus $r(T_n, T_n^0) = 2n - 8$ for odd n . This completes the proof.

Theorem 3.2. *Let $n \in \mathbb{N}$ with $n > 22$. Then $r(T_n^3, T_n'') = r(T_n^3, T_n''') = r(T_n^3, T_n^3) = 2n - 8$.*

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. When n is odd, using Lemma 2.3(i) we see that $r(T_n^3, T_n) \geq n - 4 + n - 4 = 2n - 8$. When n is even, we may construct a regular graph H with degree $n - 10$ and $V(H) = \{v_1, \dots, v_{n-6}\}$. Let G_0 be a graph given by $V(G_0) = \{v_0, v_1, \dots, v_{n-4}, u_1, \dots, u_{n-6}\}$ and

$$\begin{aligned} E(G_0) = E(H) \cup \{ & v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-5}, \dots, v_{n-6}v_{n-5}, v_1v_{n-4}, \dots, v_{n-5}v_{n-4}, \\ & v_1u_1, v_1u_2, v_2u_1, v_2u_2, \dots, v_{n-7}u_{n-7}, v_{n-7}u_{n-6}, v_{n-6}u_{n-7}, v_{n-6}u_{n-6}, \\ & u_1u_2, \dots, u_1u_{n-6}, u_2u_3, \dots, u_2u_{n-6}, u_3u_{n-6}, \dots, u_{n-7}u_{n-6} \}. \end{aligned}$$

Then $d(v_0) = d(v_{n-5}) = d(v_{n-4}) = n - 4$ and $d(v_1) = \dots = d(v_{n-6}) = d(u_1) = \dots = d(u_{n-6}) = n - 5$. Clearly $|V(G_0)| = 2n - 9$ and G_0 does not contain any copies of T_n^3 . As $\Delta(\overline{G_0}) = n - 5$, we see that $\overline{G_0}$ does not contain any copies of T_n . Thus, $r(T_n^3, T_n) \geq |V(G_0)| + 1 = 2n - 8$.

By Lemma 2.7, $\text{ex}(2n - 8; T_n'') = \text{ex}(2n - 8; T_n''') = n^2 - 9n + 29$. By Lemma 2.8, $\text{ex}(2n - 8; T_n^3) = n^2 - 9n + 29 + \max\{0, \lfloor \frac{n-37}{4} \rfloor\}$. Thus,

$$\begin{aligned} \text{ex}(2n - 8; T_n^3) + \text{ex}(2n - 8; T_n) & \leq 2n^2 - 18n + 58 + 2 \max\{0, \lfloor \frac{n-37}{4} \rfloor\} \\ & < 2n^2 - 18n + 58 + n - 22 = \binom{2n - 8}{2}. \end{aligned}$$

Hence, applying Lemma 2.1 we get $r(T_n^3, T_n) \leq 2n - 8$ and so $r(T_n^3, T_n) = 2n - 8$ as claimed.

Theorem 3.3. *Let $n \in \mathbb{N}$ with $n \geq 10$. Then $r(T_n'', T_n') = r(T_n''', T_n') = r(T_n^3, T_n') = 2n - 5$.*

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. As $\Delta(T_n) = n - 4$ and $\Delta(T_n') = n - 2$, using Lemma 2.3(ii) we see that $r(T_n, T_n') \geq 2(n - 2) - 1 = 2n - 5$. By Lemmas 2.5, 2.7 and 2.9, $\text{ex}(2n - 5; T_n') = \lfloor \frac{(n-2)(2n-6) - (n-3)}{2} \rfloor$ and $\text{ex}(2n - 5; T_n) = \frac{(n-2)(2n-5) - 3(n-4)}{2}$. Thus,

$$\begin{aligned} \text{ex}(2n - 5; T_n) + \text{ex}(2n - 5; T_n') & = \frac{(n-2)(2n-5) - 3(n-4)}{2} + \lfloor \frac{(n-2)(2n-6) - (n-3)}{2} \rfloor = \lfloor \frac{4n^2 - 23n + 37}{2} \rfloor \end{aligned}$$

$$< \frac{4n^2 - 22n + 30}{2} = \binom{2n-5}{2}.$$

Hence, $r(T_n, T'_n) \leq 2n - 5$ by Lemma 2.1. Therefore, $r(T_n, T'_n) = 2n - 5$ as claimed.

Theorem 3.4. *Let $n \in \mathbb{N}$, $n > 16$ and $i \in \{1, 2\}$. Then $r(T''_n, T_n^i) = r(T'''_n, T_n^i) = r(T_n^3, T_n^i) = 2n - 7$.*

Proof. Let $T_n \in \{T''_n, T'''_n, T_n^3\}$. As $\Delta(T_n) = n - 4$ and $\Delta(T_n^i) = n - 3$, using Lemma 2.3(ii) we see that $r(T_n, T'_n) \geq 2(n - 3) - 1 = 2n - 7$. By Lemmas 2.6, 2.7 and 2.9, $\text{ex}(2n - 7; T_n) = \frac{(n-2)(2n-7)-5(n-6)}{2}$ and $\text{ex}(2n - 7; T_n^i) = \lceil \frac{(n-2)(2n-7)}{2} \rceil - (2n - 7)$. Thus,

$$\begin{aligned} & \text{ex}(2n - 7; T_n) + \text{ex}(2n - 7; T_n^i) \\ &= \frac{(n-2)(2n-7)-5(n-6)}{2} + \lceil \frac{(n-2)(2n-7)}{2} \rceil - (2n - 7) = \lceil \frac{4n^2 - 31n + 72}{2} \rceil \\ &< \frac{4n^2 - 30n + 56}{2} = \binom{2n-7}{2}. \end{aligned}$$

Hence, $r(T_n, T_n^i) \leq 2n - 7$ by Lemma 2.1. Therefore, $r(T_n, T'_n) = 2n - 7$ as claimed.

Theorem 3.5. *Let $n \in \mathbb{N}$ with $n \geq 10$. Then $r(T_n, T_n^*) = 2n - 5$ for $T_n \in \{T''_n, T'''_n, T_n^3\}$.*

Proof. By Lemmas 2.7 and 2.9, $\text{ex}(2n - 5; T_n) = \frac{(n-2)(2n-5)-3(n-4)}{2} = n^2 - 6n + 11 < n^2 - 5n + 4$. Thus the result follows from [S, Lemma 3.1].

Remark 3.1 By [S, Theorem 6.3], we have $r(T_n, K_{1,n-1}) = 2n - 3$ for $n \geq 6$ and $T_n \in \{T''_n, T'''_n, T_n^3\}$.

Theorem 3.6. *Let $n \in \mathbb{N}$. Then $r(P_n, T''_n) = r(P_n, T'''_n) = 2n - 9$ for $n \geq 30$ and $r(P_n, T_n^3) = 2n - 9$ for $n \geq 33$.*

Proof. Suppose $n \geq 30$ and $T_n \in \{T''_n, T'''_n, T_n^3\}$. As $\Delta(T_n) = n - 4$ and $\Delta(P_n) = 2$, by Lemma 2.3(ii) we have $r(P_n, T_n) \geq 2(n - 4) - 1 = 2n - 9$. By (2.1) and Lemma 2.7, for $T_n \in \{T''_n, T'''_n\}$ we have

$$\begin{aligned} & \text{ex}(2n - 9; P_n) + \text{ex}(2n - 9; T_n) \\ &= \frac{(n-2)(2n-9)-7(n-8)}{2} + \lceil \frac{(2n-9)(n-5)}{2} \rceil = \lceil \frac{4n^2 - 39n + 119}{2} \rceil \\ &< \frac{4n^2 - 38n + 90}{2} = \binom{2n-9}{2}. \end{aligned}$$

Hence applying Lemma 2.1 we see that $r(P_n, T_n) \leq 2n - 9$ and so $r(P_n, T_n) = 2n - 9$.

Now assume $n \geq 33$. From Lemma 2.8 we have $\text{ex}(2n - 9; T_n^3) = n^2 - 10n + 24 + \lceil \frac{n}{2} \rceil$. Thus

$$\begin{aligned} & \text{ex}(2n - 9; P_n) + \text{ex}(2n - 9; T_n^3) \\ &= \frac{(n-2)(2n-9)-7(n-8)}{2} + n^2 - 10n + 24 + \lceil \frac{n}{2} \rceil = 2n^2 - 20n + 61 + \lceil \frac{n}{2} \rceil \\ &< 2n^2 - 19n + 45 = \binom{2n-9}{2}. \end{aligned}$$

Hence applying Lemma 2.1 we see that $r(P_n, T_n^3) \leq 2n - 9$ and so $r(P_n, T_n^3) = 2n - 9$.

4. Formulas for $r(K_{1,m-1}, T_n)$

Theorem 4.1. *Let $m, n_1, n_2 \in \mathbb{N}$ with $n_1 \geq m - 2 \geq n_2 \geq 2$ and $2 \mid mn_1$. If $n_1 > m - 5 + n_2 + \frac{(n_2-1)(n_2-2)}{m-1-n_2}$, then*

$$r(K_{1,m-1}, S(n_1, n_2)) = m + n_1.$$

Proof. Since $\Delta(S(n_1, n_2)) = n_1 + 1$, from Lemma 2.3(i) we see that

$$r(K_{1,m-1}, S(n_1, n_2)) \geq m - 1 + n_1 + 1 - \frac{1 - (-1)^{(m-2)n_1}}{2} = m + n_1.$$

Now we show that $r(K_{1,m-1}, S(n_1, n_2)) \leq m + n_1$. Let G be a graph of order $m + n_1$ such that \overline{G} does not contain any copies of $K_{1,m-1}$. We need to show that G contains a copy of $S(n_1, n_2)$. Clearly $\Delta(\overline{G}) \leq m - 2$ and so $\delta(G) \geq m + n_1 - 1 - (m - 2) = n_1 + 1$. Suppose $\Delta(G) = n_1 + 1 + c$, $v_0 \in V(G)$, $d(v_0) = \Delta(G)$, $\Gamma(v_0) = \{v_1, \dots, v_{n_1+1+c}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V'_1 = V(G) - V_1$. Then $|V'_1| = m - 2 - c$. As $d(v_i) \geq \delta(G) \geq n_1 + 1$ we see that

$$|\Gamma(v_i) \cap \Gamma(v_0)| \geq n_1 + 1 - 1 - |V'_1| = n_1 - (m - 2) + c \geq c.$$

Hence G contains a copy of $S(n_1, n_2)$ for $c \geq n_2$.

Now we assume $c < n_2$ and $V'_1 = \{u_1, \dots, u_{m-2-c}\}$. As $d(u_i) \geq n_1 + 1$ we see that $|\Gamma(u_i) \cap \Gamma(v_0)| \geq n_1 + 1 - (m - 3 - c)$ and so $e(V_1 V'_1) \geq (m - 2 - c)(n_1 - (m - 4 - c))$. Since $n_1 > m - 5 + n_2 + \frac{(n_2-1)(n_2-2)}{m-1-n_2}$ we know that $n_1 > m - 5 + n_2 - 2c + \frac{(n_2-2)(n_2-1-c)}{m-1-n_2}$ and so $(m - 1 - n_2)n_1 > (m - 2 - c)(m - 4 - c) + (c + 1)(n_2 - c - 1)$. Hence $e(V_1 V'_1) \geq (m - 2 - c)(n_1 - (m - 4 - c)) > (n_1 + 1 + c)(n_2 - c - 1)$. Therefore $|\Gamma(v_i) \cap V'_1| \geq n_2 - c$ for some $v_i \in \Gamma(v_0)$. From the above we know that $|\Gamma(v_i) \cap \Gamma(v_0)| \geq c$. Thus, G contains a copy of $S(n_1, n_2)$. Therefore $r(K_{1,m-1}, S(n_1, n_2)) \leq m + n_1$ and so the theorem is proved.

Corollary 4.1. *Let $m, n \in \mathbb{N}$, $n - 2 \geq m \geq 4$ and $2 \mid mn$. Then $r(K_{1,m-1}, T_n^2) = m + n - 4$.*

Proof. Since $T_n^2 = S(n - 4, 2)$, putting $n_1 = n - 4$ and $n_2 = 2$ in Theorem 4.1 we deduce the result.

Corollary 4.2. *Let $m, n \in \mathbb{N}$, $m \geq 5$, $n > m + 3 + \frac{2}{m-4}$ and $2 \mid m(n - 1)$. Then $r(K_{1,m-1}, T_n^3) = m + n - 5$.*

Proof. Since $T_n^3 = S(n - 5, 3)$, putting $n_1 = n - 5$ and $n_2 = 3$ in Theorem 4.1 we deduce the result.

Theorem 4.2. *Let $m, n_1, n_2 \in \mathbb{N}$, $n_1 \geq m - 2 > n_2$ and $2 \nmid mn_1$. If $n_1 > m - 5 + n_2 + \frac{(n_2-1)^2}{m-2-n_2}$, then*

$$r(K_{1,m-1}, S(n_1, n_2)) = m - 1 + n_1.$$

Proof. Since $\Delta(S(n_1, n_2)) = n_1 + 1$, from Lemma 2.3(i) we see that

$$r(K_{1,m-1}, S(n_1, n_2)) \geq m - 1 + n_1 + 1 - \frac{1 - (-1)^{(m-2)n_1}}{2} = m - 1 + n_1.$$

Now we show that $r(K_{1,m-1}, S(n_1, n_2)) \leq m - 1 + n_1$. Let G be a graph of order $m - 1 + n_1$ such that \overline{G} does not contain any copies of $K_{1,m-1}$. We need to show that G contains a copy of $S(n_1, n_2)$. Clearly $\Delta(\overline{G}) \leq m - 2$ and so $\delta(G) \geq m - 2 + n_1 - (m - 2) = n_1$. Since $2 \nmid mn_1$, using Euler's theorem we see that there is no regular graph of order $m - 1 + n_1$

with degree n_1 . Hence $\Delta(G) > n_1$. Suppose $\Delta(G) = n_1 + 1 + c$, $v_0 \in V(G)$, $d(v_0) = \Delta(G)$, $\Gamma(v_0) = \{v_1, \dots, v_{n_1+1+c}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V'_1 = V(G) - V_1$. Then $|V'_1| = m - 3 - c$. As $d(v_i) \geq \delta(G) \geq n_1$ we see that $|\Gamma(v_i) \cap \Gamma(v_0)| \geq n_1 - 1 - |V'_1| = n_1 - (m - 2) + c \geq c$. Hence G contains a copy of $S(n_1, n_2)$ for $c \geq n_2$.

Now we assume $c < n_2$ and $V'_1 = \{u_1, \dots, u_{m-3-c}\}$. As $d(u_i) \geq n_1$ we see that $|\Gamma(u_i) \cap \Gamma(v_0)| \geq n_1 - (m - 4 - c)$ and so $e(V_1 V'_1) \geq (m - 3 - c)(n_1 - (m - 4 - c))$. Since $n_1 > m - 5 + n_2 + \frac{(n_2-1)^2}{m-2-n_2}$ we know that $n_1 > m - 5 + n_2 - 2c + \frac{(n_2-1)(n_2-1-c)}{m-2-n_2}$ and so $(m - 2 - n_2)n_1 > (m - 3 - c)(m - 4 - c) + (c + 1)(n_2 - c - 1)$. Hence $e(V_1 V'_1) \geq (m - 3 - c)(n_1 - (m - 4 - c)) > (n_1 + 1 + c)(n_2 - c - 1)$. Therefore $|\Gamma(v_i) \cap V'_1| \geq n_2 - c$ for some $v_i \in \Gamma(v_0)$. From the above we know that $|\Gamma(v_i) \cap \Gamma(v_0)| \geq c$. Thus, G contains a copy of $S(n_1, n_2)$. Therefore $r(K_{1,m-1}, S(n_1, n_2)) \leq m - 1 + n_1$ and so the theorem is proved.

Corollary 4.3. *Let $m, n \in \mathbb{N}$, $m \geq 5$, $n \geq m + 2 + \lceil \frac{5}{m} \rceil$ and $2 \nmid mn$. Then $r(K_{1,m-1}, T_n^2) = m + n - 5$.*

Proof. Since $T_n^2 = S(n - 4, 2)$, putting $n_1 = n - 4$ and $n_2 = 2$ in Theorem 4.2 we deduce the result.

Corollary 4.4. *Let $m, n \in \mathbb{N}$, $m \geq 6$, $n > m + 3 + \frac{4}{m-5}$ and $2 \nmid m(n - 1)$. Then $r(K_{1,m-1}, T_n^3) = m + n - 6$.*

Proof. Since $T_n^3 = S(n - 5, 3)$, putting $n_1 = n - 5$ and $n_2 = 3$ in Theorem 4.2 we deduce the result.

Theorem 4.3. *Let $m, n \in \mathbb{N}$, $n > m \geq 6$ and $2 \nmid mn$. Then*

$$r(K_{1,m-1}, T_n^1) = r(K_{1,m-1}, T_n^2) = m + n - 5.$$

Proof. As $2 \nmid mn$ we have $n \geq m + 2$. Let $j \in \{1, 2\}$ and let G be a graph of order $m + n - 5$ such that \overline{G} does not contain any copies of $K_{1,m-1}$. We show that G contains a copy of T_n^j . Clearly $\Delta(\overline{G}) \leq m - 2$ and so $\delta(G) \geq m + n - 6 - (m - 2) = n - 4$. If $\Delta(G) = n - 4$, then G is a regular graph of order $m + n - 5$ with degree $n - 4$ and so $(m + n - 5)(n - 4) = 2e(G)$. Since $m + n - 5$ and $n - 4$ are odd, we get a contradiction. Thus, $\Delta(G) \geq n - 3$. Assume $v_0 \in V(G)$, $d(v_0) = \Delta(G) = n - 3 + c$, $\Gamma(v_0) = \{v_1, \dots, v_{n-3+c}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V'_1 = V(G) - V_1$. Since $|V'_1| = m - 3 - c$ and $\delta(G) \geq n - 4$ we see that for $G' = G(\Gamma(v_0))$ and $v_i \in \Gamma(v_0)$, $d_{G'}(v_i) \geq n - 4 - 1 - (m - 3 - c) = n - m - 2 + c \geq c$. Hence, for $c \geq 3$ we have $d_{G'}(v_i) \geq 3$ and so G contains a copy of T_n^j .

Now assume $c \leq 2$. As $m > 5$ we have $|V'_1| = m - 3 - c \geq m - 5 \geq 1$. Since $m - 4 - c < n - 4$ and $\delta(G) \geq n - 4$ we see that for any $v \in V'_1$, $|\Gamma(v) \cap \{v_1, \dots, v_{n-3+c}\}| \geq n - 4 - (m - 4 - c) = n - m + c \geq 2$. Suppose $V'_1 = \Gamma_2(v_0) = \{u_1, \dots, u_{m-3-c}\}$ and $u_1 v_1 \in E(G)$. Since $\delta(G) \geq n - 4 \geq m - 2 \geq 4$, we see that $d(v_1) \geq 4$ and hence G contains a copy of T_n^j .

Assume $c = 1$. As $|V'_1| = m - 4 \geq 2$ and $\delta(G) \geq n - 4$ we see that $|\Gamma(v) \cap \{v_1, \dots, v_{n-2}\}| \geq n - 4 - (m - 5) = n - m + 1 \geq 3$ for $v \in V'_1$ and so G contains a copy of T_n^j .

Assume $c = 0$. Then $|V'_1| = m - 3 \geq 3$. As $\delta(G) \geq n - 4$ we see that $|\Gamma(v) \cap \{v_1, \dots, v_{n-3}\}| \geq n - 4 - (m - 4) = n - m \geq 2$ for $v \in V'_1$. Hence $e(V_1 V'_1) \geq (m - 3)(n - m)$. Since $n \geq m + 2$ and $m > 5$ we see that $(m - 4)n \geq (m - 4)(m + 2) = m^2 - 2m - 8 > m^2 - 3m - 3$ and so $(m - 3)(n - m) > n - 3$. Hence $e(V_1 V'_1) > n - 3$ and so $|\Gamma(v_i) \cap V'_1| \geq 2$ for some $i \in \{1, 2, \dots, n - 3\}$. Therefore G contains a copy of T_n^j .

By the above, G contains a copy of T_n^j . Therefore $r(K_{1,m-1}, T_n^j) \leq m + n - 5$. By Lemma 2.3, $r(K_{1,m-1}, T_n^j) \geq m - 1 + n - 3 - \frac{1 - (-1)^{(m-2)(n-4)}}{2} = m + n - 5$. Hence $r(K_{1,m-1}, T_n^j) = m + n - 5$ as claimed.

Lemma 4.1. *Let $m, n \in \mathbb{N}$, $n \geq 15$, $m \geq 7$, $n > m + 1 + \frac{8}{m-6}$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Let G_m be a connected graph of order m such that $ex(m + n - 5; G_m) \leq \frac{(m-2)(m+n-5)}{2}$. Then $r(G_m, T_n) \leq m + n - 5$. Moreover, if $m - 1 \mid n - 5$, then $r(G_m, T_n) = m + n - 5$.*

Proof. If $T_n \neq T_n^3$ or $m \notin \{n - 3, n - 4\}$, by Lemmas 2.7 and 2.9 we have

$$\begin{aligned} & ex(m + n - 5; T_n) \\ &= \frac{(n - 2)(m + n - 5) - (m - 4)(n - m + 3)}{2} + \max \left\{ 0, \left\lceil \frac{(m - 4)(n - m) - 3(n - 1)}{2} \right\rceil \right\} \\ &= \frac{(n - 2)(m + n - 5) - (m - 4)(n - m + 3)}{2} + \max \left\{ 0, \left\lceil \frac{(m - 7)n - (m^2 - 4m - 3)}{2} \right\rceil \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} & ex(m + n - 5; G_m) + ex(m + n - 5; T_n) \\ &\leq \frac{(m - 2)(m + n - 5)}{2} + \frac{(n - 2)(m + n - 5) - (m - 4)(n - m + 3)}{2} \\ &\quad + \max \left\{ 0, \left\lceil \frac{(m - 7)n - (m^2 - 4m - 3)}{2} \right\rceil \right\}. \end{aligned}$$

If $(m - 7)n \geq m^2 - 4m - 3$, then

$$\begin{aligned} & ex(m + n - 5; G_m) + ex(m + n - 5; T_n) \\ &\leq \frac{(m + n - 4)(m + n - 5) - 3(m + n - 5)}{2} < \binom{m + n - 5}{2}. \end{aligned}$$

If $(m - 7)n < m^2 - 4m - 3$, then $m = 7$ or $n < m + 3 + \frac{18}{m-7}$. Since $n > m + 1 + \frac{8}{m-6}$, we see that $(m - 6)n > m^2 - 5m + 2$, $(m - 4)(n - m + 3) > 2(m + n - 5)$ and so

$$\begin{aligned} & ex(m + n - 5; G_m) + ex(m + n - 5; T_n) \\ &\leq \frac{(m + n - 4)(m + n - 5) - (n - m + 3)(m - 4)}{2} < \binom{m + n - 5}{2}. \end{aligned}$$

Hence, using Lemma 2.1 we always have $r(G_m, T_n) \leq m + n - 5$.

For $m = n - 3$, using Lemma 2.8 we see that

$$\begin{aligned} & ex(m + n - 5; G_m) + ex(m + n - 5; T_n^3) \\ &= ex(2n - 8; G_{n-3}) + ex(2n - 8; T_n^3) \\ &\leq \frac{(2n - 8)(n - 5)}{2} + n^2 - 9n + 29 + \max \left\{ 0, \left\lceil \frac{n - 37}{4} \right\rceil \right\} \\ &= 2n^2 - 18n + 49 + \max \left\{ 0, \left\lceil \frac{n - 37}{4} \right\rceil \right\} < 2n^2 - 17n + 36 = \binom{m + n - 5}{2}. \end{aligned}$$

For $m = n - 4$, using Lemma 2.8 we see that

$$ex(m + n - 5; G_m) + ex(m + n - 5; T_n^3)$$

$$\begin{aligned}
&= \text{ex}(2n-9; G_{n-4}) + \text{ex}(2n-9; T_n^3) \\
&\leq \frac{(2n-9)(n-6)}{2} + n^2 - 10n + 24 + \max\left\{\left\lceil \frac{n}{2} \right\rceil, 13\right\} \\
&= 2n^2 - 21n + 51 + \frac{n}{2} + \max\left\{\left\lceil \frac{n}{2} \right\rceil, 13\right\} < 2n^2 - 19n + 45 = \binom{m+n-5}{2}.
\end{aligned}$$

Thus, using Lemma 2.1 we have $r(G_m, T_n^3) \leq m+n-5$ for $m = n-4, n-3$.

Now assume $m-1 \mid n-5$. Then $m+n-6 = k(m-1)$ for $k \in \{2, 3, \dots\}$. Since $\Delta(\overline{kK_{m-1}}) = n-5$ we see that kK_{m-1} does not contain G_m as a subgraph and $\overline{kK_{m-1}}$ does not contain T_n as a subgraph. Hence $r(G_m, T_n) > k(m-1) = m+n-6$ and so $r(G_m, T_n) = m+n-5$. The proof is now complete.

Theorem 4.4. *Let $m, n \in \mathbb{N}$, $n \geq 15$, $m \geq 7$, $n > m+1 + \frac{8}{m-6}$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. If $2 \mid m(n-1)$, then $r(K_{1,m-1}, T_n) = m+n-5$.*

Proof. By Euler's theorem or Lemma 2.4, $\text{ex}(m+n-5; K_{1,m-1}) \leq \frac{(m-2)(m+n-5)}{2}$. Thus, applying Lemma 4.1 we obtain $r(K_{1,m-1}, T_n) \leq m+n-5$. Suppose $2 \nmid m(n-1)$. By Lemma 2.3,

$$r(K_{1,m-1}, T_n) \geq m-1 + n-4 - \frac{1 - (-1)^{(m-2)(n-5)}}{2} = m+n-5.$$

Thus the result follows.

Corollary 4.5. *Let $n \in \mathbb{N}$, $n \geq 17$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Then $r(K_{1,n-3}, T_n) = 2n-7$.*

Proof. Taking $m = n-2$ in Theorem 4.4 we obtain the result.

Theorem 4.5. *Let $m, n \in \mathbb{N}$, $m \geq 6$, $n \geq m+3$ and $2 \nmid m(n-1)$. Then*

$$r(K_{1,m-1}, T_n'') = r(K_{1,m-1}, T_n''') = m+n-6.$$

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$, and let G be a graph of order $m+n-6$ such that \overline{G} does not contain any copies of $K_{1,m-1}$. Clearly $\Delta(\overline{G}) \leq m-2$ and so $\delta(G) \geq m+n-7-(m-2) = n-5$. If $\Delta(G) = n-5$, then G is a regular graph of order $m+n-6$ with degree $n-5$ and so $(m+n-6)(n-5) = 2e(G)$. Since $m+n-6$ and $n-5$ are odd, we get a contradiction. Thus, $\Delta(G) \geq n-4$. Assume $v_0 \in V(G)$, $d(v_0) = \Delta(G) = n-4+c$, $\Gamma(v_0) = \{v_1, \dots, v_{n-4+c}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V_1' = V(G) - V_1$. Since $|V_1'| = m-3-c$ and $\delta(G) \geq n-5$ we see that for $G' = G(\Gamma(v_0))$ and $v_i \in \Gamma(v_0)$, $d_{G'}(v_i) \geq n-5-1-(m-3-c) = n-m-3+c \geq c$. Hence, G contains a copy of T_n for $c \geq 3$.

Now assume $c \leq 2$. Since $|V_1'| = m-3-c$ and $\delta(G) \geq n-5$ we see that for any $v \in V_1'$, $|\Gamma(v) \cap \{v_1, \dots, v_{n-4+c}\}| \geq n-5-(m-4-c) = n-m-1+c \geq c+2$. Suppose $V_1' = \Gamma_2(v_0) = \{u_1, \dots, u_{m-3-c}\}$ and $u_1v_1 \in E(G)$. Since $d(v_1) \geq n-5 \geq 4$, for $c=2$ we see that $|\Gamma(v_1) \cap \{v_2, \dots, v_{n-2}, u_1, \dots, u_{m-3-c}\}| \geq 3$ and so G contains a copy of T_n .

From now on suppose $T_n \in \{T_n'', T_n'''\}$. Assume $c=1$. Then $|V_1'| = m-4 \geq 2$ and $|\Gamma(u_i) \cap \{v_1, \dots, v_{n-3}\}| \geq c+2 = 3$ for $i=1, 2$. As $u_1v_1 \in E(G)$ we see that G contains a copy of T_n . Now assume $c=0$. Since $|V_1'| = m-3 \geq 3$ and $\delta(G) \geq n-5$ we see that $|\Gamma(v) \cap \{v_1, \dots, v_{n-4}\}| \geq n-5-(m-4) = n-m-1 \geq 2$ for $v \in V_1'$ and so $e(V_1V_1') \geq (m-3)(n-m-1)$. Since $n \geq m+3$ we see that $(m-4)n \geq (m-4)(m+3) = m^2 - m - 12 > m^2 - 2m - 7$ and so $e(V_1V_1') \geq (m-3)(n-m-1) > n-4$. Therefore, $|\Gamma(v_i) \cap V_1'| \geq 2$ for some $i \in \{1, 2, \dots, n-4\}$ and thus G contains a copy of T_n .

By the above, G contains a copy of T_n . Hence $r(K_{1,m-1}, T_n) \leq m + n - 6$. By Lemma 2.3, $r(K_{1,m-1}, T_n) \geq m - 1 + n - 4 - \frac{1 - (-1)^{(m-2)(n-5)}}{2} = m + n - 6$. Thus $r(K_{1,m-1}, T_n) = m + n - 6$ as asserted.

Theorem 4.6. *Let $n \in \mathbb{N}$ with $n \geq 15$. Then $r(K_{1,n-4}, T_n^3) = 2n - 8$.*

Proof. By Lemmas 2.4 and 4.1, $r(K_{1,n-4}, T_n^3) \leq 2n - 8$. If $2 \nmid n$, from Lemma 2.3 we have $r(K_{1,n-4}, T_n^3) \geq n - 4 + n - 4 = 2n - 8$. Thus the result is true for odd n . Now assume $2 \mid n$. Let G_0 be the graph of order $2n - 9$ constructed in Theorem 3.2. Then G_0 does not contain T_n^3 as a subgraph. As $\delta(G_0) = n - 5$ we have $\Delta(\overline{G_0}) = 2n - 10 - (n - 5) = n - 5$ and so $\overline{G_0}$ does not contain $K_{1,n-4}$ as a subgraph. Hence $r(K_{1,n-4}, T_n^3) > |V(G_0)| = 2n - 9$ and so $r(K_{1,n-4}, T_n^3) = 2n - 8$. This completes the proof.

Theorem 4.7. *Let $n \in \mathbb{N}$ with $n \geq 10$. Then*

$$r(K_{1,n-2}, T_n^3) = r(K_{1,n-2}, T_n'') = r(K_{1,n-2}, T_n''') = 2n - 5.$$

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. Since $\Delta(K_{1,n-2}) = n - 2$ and $\Delta(T_n) = n - 4$, by Lemma 2.3(ii) we have $r(K_{1,n-2}, T_n) \geq 2(n - 2) - 1 = 2n - 5$. By Lemmas 2.4, 2.7 and 2.9, $\text{ex}(2n - 5; K_{1,n-2}) = \lfloor \frac{(n-3)(2n-5)}{2} \rfloor = n^2 - 6n + 8 + \lfloor \frac{n-1}{2} \rfloor$ and $\text{ex}(2n - 5; T_n) = \frac{(n-2)(2n-5) - 3(n-4)}{2} = n^2 - 6n + 11$. Thus,

$$\begin{aligned} & \text{ex}(2n - 5; K_{1,n-2}) + \text{ex}(2n - 5; T_n) \\ &= n^2 - 6n + 8 + \lfloor \frac{n-1}{2} \rfloor + n^2 - 6n + 11 < 2n^2 - 11n + 15 = \binom{2n-5}{2}. \end{aligned}$$

Now applying Lemma 2.1 we get $r(K_{1,n-2}, T_n) \leq 2n - 5$ and so $r(K_{1,n-2}, T_n) = 2n - 5$ as claimed.

5. Formulas for $r(T'_m, T''_n), r(T'_m, T'''_n)$ and $r(T'_m, T_n^3)$

Theorem 5.1. *Let $m, n \in \mathbb{N}$, $n \geq 15$, $m \geq 9$ and $m - 1 \mid n - 5$. Suppose $G_m \in \{P_m, T'_m, T_m^*, T_m^1, T_m^2, T_m^3, T_m'', T_m'''\}$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Then $r(G_m, T_n) = m + n - 5$.*

Proof. By (2.1) and Lemmas 2.5-2.12, $\text{ex}(m + n - 5; G_m) \leq \frac{(m-2)(m+n-5)}{2}$. Thus applying Lemma 4.1 we deduce the result.

Theorem 5.2. *Let $m, n \in \mathbb{N}$, $m \geq 9$ and $n > m + 1 + \max\{3, \frac{11}{m-8}\}$. Suppose $m - 1 \nmid n - 5$. Then*

$$r(T'_m, T_n'') = (T'_m, T_n''') = r(T'_m, T_n^3) = m + n - 6.$$

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. Since $\Delta(T'_m) = m - 2 < n - 4 = \Delta(T_n)$, by Lemma 2.3 we have $r(T'_m, T_n) \geq m - 2 + n - 4 = m + n - 6$. Note that $n \geq 15$ and $3 \leq m - 5 \leq n - 9$. From Lemmas 2.7 and 2.9 we see that if $(m - 5)(n - m + 1) \geq 3(n - 1)$, then

$$\begin{aligned} & \text{ex}(m + n - 6; T_n) \\ &= \frac{(n - 2)(m + n - 6) - (m - 5)(n - m + 4)}{2} + \left\lceil \frac{(m - 5)(n - m + 1) - 3(n - 1)}{2} \right\rceil \\ &\leq \frac{(n - 2)(m + n - 6) - (m - 5)(n - m + 4)}{2} + \frac{(m - 5)(n - m + 1) - 3(n - 1)}{2} \end{aligned}$$

$$= \frac{(n-5)(m+n-6)}{2}.$$

Since $n > m+1 + \frac{11}{m-8}$ we see that $(m-8)n > m^2 - 7m + 3$ and so $(m-5)(n-m+4) > 3(m+n-6) - (m-1)$. Hence, if $(m-5)(n-m+1) < 3(n-1)$, using Lemmas 2.7 and 2.9 we have

$$\begin{aligned} & \text{ex}(m+n-6; T_n) \\ &= \frac{(n-2)(m+n-6) - (m-5)(n-m+4)}{2} \\ &< \frac{(n-2)(m+n-6) - 3(m+n-6) + m-1}{2} = \frac{(n-5)(m+n-6) + m-1}{2}. \end{aligned}$$

Recall that $m-1 \nmid n-5$. From Lemma 2.5 we know that $\text{ex}(m+n-6; T'_m) \leq \frac{(m-2)(m+n-6) - (m-1)}{2}$. Thus, applying the above we deduce that

$$\begin{aligned} & \text{ex}(m+n-6; T'_m) + \text{ex}(m+n-6; T_n) \\ &< \frac{(n-5)(m+n-6) + m-1}{2} + \frac{(m-2)(m+n-6) - (m-1)}{2} = \binom{m+n-6}{2}. \end{aligned}$$

Hence applying Lemma 2.1 we see that $r(T'_m, T_n) \leq m+n-6$ and so $r(T'_m, T_n) = m+n-6$ as claimed.

Theorem 5.3. *Let $n \in \mathbb{N}$. Then $r(T'_{n-3}, T''_n) = r(T'_{n-3}, T'''_n) = 2n-9$ for $n \geq 10$, and $r(T'_{n-3}, T_n^3) = 2n-9$ for $n \geq 15$.*

Proof. Suppose $n \geq 10$ and $T_n \in \{T''_n, T'''_n, T_n^3\}$. Since $\Delta(T_n) = n-4 > n-5 = \Delta(T'_{n-3})$, from Lemma 2.3(ii) we have $r(T'_{n-3}, T_n) \geq 2(n-4) - 1 = 2n-9$. By Lemma 2.5, $\text{ex}(2n-9; T'_{n-3}) = \frac{(n-5)(2n-9)}{2} = n^2 - 10n + 25$. By Lemma 2.7, for $T_n \in \{T''_n, T'''_n\}$ we have

$$\begin{aligned} \text{ex}(2n-9; T_n) &= \frac{(n-2)(2n-9) - 7(n-8)}{2} + \max \left\{ 0, \left\lceil \frac{4(n-8) - 3(n-1)}{2} \right\rceil \right\} \\ &= \begin{cases} \left\lceil \frac{(n-5)(2n-9)}{2} \right\rceil \leq n^2 - \frac{19}{2}n + \frac{45}{2} & \text{if } n \geq 29, \\ n^2 - 10n + 37 & \text{if } n < 29 \end{cases} \\ &< n^2 - 9n + 30 \end{aligned}$$

and so

$$\text{ex}(2n-9; T'_{n-3}) + \text{ex}(2n-9; T_n) < n^2 - 10n + 25 + n^2 - 9n + 30 = \binom{2n-9}{2}.$$

Now applying Lemma 2.1 we obtain $r(T'_{n-3}, T_n) \leq 2n-9$ and so $r(T'_{n-3}, T_n) = 2n-9$. For $n \geq 15$, from Lemma 2.8 we have

$$\text{ex}(2n-9; T_n^3) = n^2 - 10n + 24 + \max \left\{ \left\lceil \frac{n}{2} \right\rceil, 13 \right\} < n^2 - 9n + 30.$$

Thus, $\text{ex}(2n-9; T'_{n-3}) + \text{ex}(2n-9; T_n^3) < \binom{2n-9}{2}$. Applying Lemma 2.1 we obtain $r(T'_{n-3}, T_n^3) \leq 2n-9$ and so $r(T'_{n-3}, T_n^3) = 2n-9$, which completes the proof.

Theorem 5.4. *Let $m, n \in \mathbb{N}$ with $n > m \geq 10$, and $T_m \in \{T''_m, T'''_m, T_m^3\}$. Then*

$$r(T_m, T'_n) = r(T_m, T_n^*) = \begin{cases} m+n-3 & \text{if } m-1 \mid n-3, \\ m+n-4 & \text{if } m-1 \nmid n-3 \text{ and } n \geq (m-3)^2 + 2. \end{cases}$$

Proof. If $m - 1 \mid n - 3$, from Lemmas 2.7 and 2.9 we know that $\text{ex}(m + n - 3; T_m) = \frac{(m-2)(m+n-3)}{2}$. Thus, the result follows from [S, Theorems 4.1 and 5.1].

Now assume $m - 1 \nmid n - 3$. By Lemma 2.10, $\text{ex}(m + n - 4; T_m) < \frac{(m-2)(m+n-4)}{2}$. Applying [S, Theorems 4.4 and 5.4] we deduce the result.

6. Formulas for $r(T_m^0, T_n)$ with $\Delta(T_m^0) = m - 3$ and $\Delta(T_n) = n - 4$

Lemma 6.1 ([H, Theorem 8.3, pp.11-12]). *Let $a, b, n \in \mathbb{N}$. If a is coprime to b and $n \geq (a - 1)(b - 1)$, then there are two nonnegative integers x and y such that $n = ax + by$.*

Theorem 6.1. *Let $m, n \in \mathbb{N}$ with $n \geq 15$, $m \geq 9$, $m - 1 \nmid n - 5$ and $n > m + 1 + \frac{12}{m-8}$. Suppose $T_m^0 \in \{T_m^*, T_m^1, T_m^2\}$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Then $r(T_m^0, T_n) = m + n - 7$ or $m + n - 6$. If $n \geq (m - 3)^2 + 4$ or $m + n - 7 = (m - 1)x + (m - 2)y$ for some nonnegative integers x and y , then $r(T_m^0, T_n) = m + n - 6$.*

Proof. As $\Delta(T_m^0) = m - 3 < n - 4 = \Delta(T_n)$, from Lemma 2.3 we know that $r(T_m^0, T_n) \geq m - 3 + n - 4 = m + n - 7$. Since $m - 1 \nmid n - 5$, from Lemmas 2.6, 2.11 and 2.12 we have $\text{ex}(m + n - 6; T_m^0) \leq \frac{(m-2)(m+n-6)-(m-2)}{2}$. If $(m - 5)(n - m + 1) - 3(n - 1) \geq 0$, from Lemmas 2.7 and 2.9(ii) we have

$$\begin{aligned} & \text{ex}(m + n - 6; T_n) \\ &= \frac{(n - 2)(m + n - 6) - (m - 5)(n - m + 4)}{2} + \left\lceil \frac{(m - 5)(n - m + 1) - 3(n - 1)}{2} \right\rceil \\ &= \left\lceil \frac{(n - 5)(m + n - 6)}{2} \right\rceil \leq \frac{(n - 5)(m + n - 6)}{2}. \end{aligned}$$

Hence

$$\begin{aligned} & \text{ex}(m + n - 6; T_m^0) + \text{ex}(m + n - 6; T_n) \\ &< \frac{(m - 2)(m + n - 6)}{2} + \frac{(n - 5)(m + n - 6)}{2} = \binom{m + n - 6}{2}. \end{aligned}$$

Now applying Lemma 2.1 we get $r(T_m^0, T_n) \leq m + n - 6$.

If $(m - 5)(n - m + 1) - 3(n - 1) < 0$, from Lemmas 2.7 and 2.9(ii) we have $\text{ex}(m + n - 6; T_n) = \frac{(n-2)(m+n-6)-(m-5)(n-m+4)}{2}$. Since $n > m + 1 + \frac{12}{m-8}$, we see that $(m - 8)n > m^2 - 7m + 4$ and so $(m - 5)(n - m + 4) > 3(m + n - 6) - (m - 2)$. Hence $\text{ex}(m + n - 6; T_n) = \frac{(n-2)(m+n-6)-(m-5)(n-m+4)}{2} < \frac{(n-5)(m+n-6)+m-2}{2}$ and therefore

$$\begin{aligned} & \text{ex}(m + n - 6; T_m^0) + \text{ex}(m + n - 6; T_n) \\ &< \frac{(m - 2)(m + n - 6) - (m - 2)}{2} + \frac{(n - 5)(m + n - 6) + m - 2}{2} = \binom{m + n - 6}{2}. \end{aligned}$$

Applying Lemma 2.1 we get $r(T_m^0, T_n) \leq m + n - 6$.

If $m + n - 7 = (m - 1)x + (m - 2)y$ for some nonnegative integers x and y , setting $G = xK_{m-1} \cup yK_{m-2}$ we find that G does not contain any copies of T_m^0 . Observe that

$\Delta(\overline{G}) \leq n - 5$. We see that \overline{G} does not contain any copies of T_n . Hence $r(T_m^0, T_n) > |V(G)| = m + n - 7$ and so $r(T_m^0, T_n) = m + n - 6$. If $n \geq (m - 3)^2 + 4$, then $m + n - 7 \geq (m - 2)(m - 3)$. Applying Lemma 6.1 we see that $m + n - 7 = (m - 1)x + (m - 2)y$ for some nonnegative integers x and y and so $r(T_m^0, T_n) = m + n - 6$ as claimed.

In conclusion we pose two conjectures.

Conjecture 1. *Suppose $p, n \in \mathbb{N}$ with $p \geq n \geq 3$. Let T_n be a tree on n vertices, and $\alpha_2(T_n)$ be the maximal number of vertices in the partition of the vertices of T_n when T_n is viewed as a bipartite graph. If G is a connected graph of order p and $\delta(G) \geq \alpha_2(T_n)$, then G contains a copy of T_n as a subgraph.*

It is well known that (see [CL,p.72]) if G is a graph with $\delta(G) \geq n - 1$, then G contains every tree on n vertices.

Conjecture 2. *Let $m, n \in \mathbb{N}$ with $m, n \geq 3$, and let T_n be a tree on n vertices. If $\alpha_2(T_n) \geq m - 2$, then $r(K_{1,m-1}, T_n) \leq m - 1 + \alpha_2(T_n)$.*

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